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AN ANALOGUE OF SIEGEL'S ϕ-OPERATOR FOR AUTOMORPHIC FORMS FOR $GL_n(\mathbb{Z})$

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ABSTRACT. If $\mathscr{S}P_n$ is the symmetric space of $n \times n$ positive matrices, $Y \in \mathscr{S}P_n$ can be decomposed into

$$Y = \begin{pmatrix} 1 & 0 \\ x & I \end{pmatrix} \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \begin{pmatrix} 1 & {}^T x \\ 0 & I \end{pmatrix},$$

where $W \in \mathscr{S}P_{n-1}$. By letting $v \to \infty$ we obtain the ϕ-operator that attaches to every automorphic form for $GL_n(\mathbb{Z})$, $f(Y)$, an automorphic form for $GL_{n-1}(\mathbb{Z})$, $f|\phi(W)$.

1. Introduction

The action of the modular group $SL_2(\mathbb{Z})$ on the upper half-plane of complex numbers, $z = x + iy$, $y > 0$, is well known. If we identify z with the matrix

$$\begin{pmatrix} y^{-1} & y^{-1}x \\ y^{-1}x & y^{-1}x^2 + y \end{pmatrix} = y^{-1} \begin{pmatrix} 1 & x \\ x & x^2 + y^2 \end{pmatrix}$$

we have identified the upper half-plane with the space of positive definite 2×2 real matrices of determinant one. Let $\mathscr{P}_n$ be the symmetric space of positive definite $n \times n$ real matrices (or positive definite quadratic forms). Further, let $\mathscr{S}P_n$ denote the subspace of matrices with determinant one. The case above is $n = 2$. Also, define $\Gamma_n = GL_n(\mathbb{Z})/\{\pm I\}$, with I the $n \times n$ identity matrix. With only slight modification we could have taken Γ_n to be $SL_n(\mathbb{Z})/\{\pm I\}$ for even n and $SL_n(\mathbb{Z})$ for odd n. If n is odd $SL_n(\mathbb{Z}) \cong GL_n(\mathbb{Z})/\{\pm I\}$ anyway. This paper can be followed with either choice of Γ_n.

If $\gamma \in \Gamma_n$ and $Y \in \mathscr{P}_n$, Γ_n acts on $\mathscr{P}_n$ by sending Y to ${}^T\gamma Y \gamma$, where ${}^T\gamma$ is the transpose. We will use Siegel's notation $Y[\gamma]$ for ${}^T\gamma Y \gamma$. An automorphic form for Γ_n is a function $f : \mathscr{P}_n \to C$ satisfying the following conditions:

(i) f is an eigenfunction for all the $GL_n(\mathbb{R})$-invariant differential operators.
(ii) $f(Y[\gamma]) = f(Y)$ for all $Y \in \mathscr{P}_n$ and $\gamma \in \Gamma_n$.
(iii) If $p_{-s}(Y)$ is Selberg's power function defined by $p_{-s}(Y) = \prod_{j=1}^n |Y_j|^{-s_j}$ for $s \in \mathbb{C}^n$ then there are $C > 0$ and $s \in \mathbb{C}^n$ such that $|f(Y)| \leq C|p_{-s}(Y)|$ as the upper left determinants $|Y_j| \to \infty$.

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The differential operators in condition (i) have the basis

$$L_j = \operatorname{Tr}\left(\left(Y\frac{\partial}{\partial Y}\right)^j\right), \qquad j = 1, \dots, n, \quad \frac{\partial}{\partial Y} = \begin{pmatrix} \frac{\partial}{\partial y_{ii}} & & \frac{1}{2}\frac{\partial}{\partial y_{ij}} \\ & \ddots & \\ \frac{1}{2}\frac{\partial}{\partial y_{ij}} & & \frac{\partial}{\partial y_{nn}} \end{pmatrix};$$

here L_2 is the Laplace operator on $\mathscr{P}_n$. Selberg [8] showed that integral operators could be used in place of the differential operators.

For the rest of the discussion, we will be primarily concerned with the symmetric spaces $\mathscr{S}P_n$. For $Y \in \mathscr{S}P_n$ we will use the decomposition

$$Y = \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x \\ 0 & I_{n-1} \end{bmatrix}, \tag{1}$$

$$v > 0, \quad W \in \mathscr{S}P_{n-1}, \quad x \in \mathbb{R}^{n-1}.$$

Here the square bracket notation has the same meaning as before. It may seem a little strange at first to decompose Y in this fashion, but as was shown in [3] this generalizes the identification of the upper half-plane to $\mathscr{S}P_2$ to the spaces $\mathscr{S}P_n$ in a convenient fashion. For the matters at hand, $W \in \mathscr{S}P_{n-1}$ means that for fixed v, f can be thought of as a function of W, and so a function on $\mathscr{S}P_{n-1}$.

The goal in this paper is to stick as closely as possible to the classical approach to the study of Maass wave forms. There will also be some analogies to Seigel's modular group, $Sp_n(\mathbb{Z})$, which also is a generalization of $SL_2(\mathbb{Z})$, and which acts on the generalized upper half-space $H_n = \{Z = X + iY \mid X, Y \in \mathbb{R}^{n\times n}, \ Y \in \mathscr{P}_n\}$. For more details see [7]. Most importantly, there will be an analogue of Siegel's ϕ operator which sends modular forms on $Sp_n(\mathbb{Z})$ to $Sp_{n-1}(\mathbb{Z})$. If f is an automorphic form on Γ_n there will be associated to it an automorphic form $f|\phi$ on Γ_{n-1} by letting $v \to \infty$.

There is one more analogue to Siegel's modular group worth mentioning. A fundamental domain for $\mathscr{S}P_n$ is a subset of $\mathscr{S}P_n$ which covers the whole space by the action of Γ_n without any overlap other than on the boundaries. In other words it is equivalent to the quotient space $\mathscr{S}P_n/\Gamma_n$. There are many choices of fundamental domains that could be used, but the domain developed in [3] was designed specifically to have the "right" boundaries to be most compatible with the study of automorphic forms. This fundamental domain, $\mathscr{F}_n$, is defined to be the subset of $\mathscr{S}P_n$ satisfying the following three conditions:

(i) $(a + {}^T x\, c)^2 + v^{n/(n-1)} W[c] \geq 1$ for any $a \in \mathbb{Z}$, $c \in \mathbb{Z}^{n-1}$, and $\left(\begin{smallmatrix} a & {}^T b \\ c & D \end{smallmatrix}\right) \in \Gamma_n$;

(ii) $W \in \mathscr{F}_{n-1}$;

(iii) $0 \leq x_1 \leq \frac{1}{2}$, $|x_j| \leq \frac{1}{2}$ for $j = 2, 3, \dots, n-1$.

In this definition we again used the decomposition (1).

In [3] it is also shown that $\mathscr{F}_n$ as defined has only a finite number of boundary portions, i.e., condition (i) requires only certain a and c forming the first column of a matrix in Γ_n. Having a finite number of boundary conditions is quite important if the fundamental domain is to be useful at all. Also in [3] it is seen that two of the most important features of $\mathscr{F}_n$ are the boundaries defined by condition iii) for the x_j, and the behavior as $v \to \infty$.

Since for all but x_1 the distance between boundaries of the x_j is a period of $e^{2\pi i m_j x_j}$, $\mathcal{F}_n$ will be most convenient for studying Fourier expansions. As far as the behavior as $v \to \infty$, consider condition (i) above. As $v \to \infty$, $v^{n/(n-1)} W[c] \geq 1$ as long as $W[c] > 0$. However, this is always so, as $W \in \mathcal{SP}_{n-1}$. Condition (ii) then puts $W \in \mathcal{F}_{n-1}$. As $v \to \infty$, we approach what is called a cusp of $\mathcal{F}_n$, and this cusp consists of $\mathcal{F}_{n-1}$ and its cusps. This is quite similar to the Satake compactification for the Siegel modular group. This also gives the first hint that if we take an automorphic form for Γ_n, $f(Y)$, and let $v \to \infty$, we should see something related to a form for Γ_{n-1}, as a function of W.

2. Special functions on $\mathcal{P}_n$

Maass wave forms, which are the automorphic forms defined above when $n = 2$, have Fourier expansion

$$f(z) = ay^s + by^{1-s} + \sum_{m \neq 0} a_m y^{1/2} K_{s-1/2}(2\pi |m|_y) e^{2\pi i m x}, \tag{2}$$

where λ can be written as $\lambda = s(s-1)$ if $\Delta f = \lambda f$. $K_s(y)$ is a Bessel function which can be defined as

$$K_s(y) = \frac{1}{2} \int_0^\infty t^{s-1} e^{-(y/2)[t+1/t]}\, dt. \tag{3}$$

Lebedev [5] illustrates that $K_s(y) \sim \sqrt{\pi/2y}\, e^{-y}$ as $y \to \infty$. Thus for $\mathrm{Re}(s) > 1$ in (2) above we see that $f(z) \sim ay^s$ as $y \to \infty$. We shall want to use generalizations of these Bessel functions in the Fourier expansions of automorphic forms for Γ_n to obtain similar asymptotic results. The situation for Γ_n is more complicated, however, so we will define two generalizations of the K-Bessel function. The first is a slight variant of a function studied by Bengtson [1] and Terras (as in [11, Vol. II]). Define

$$k_{n-1}(s, r, Y) = \int_{u \in \mathbb{R}^{n-1}} p_{-s}\left(Y^{-1} \begin{bmatrix} 1 & 0 \\ u & I \end{bmatrix}\right) e^{2\pi i\, {}^T r u}\, du \tag{4}$$

for $s \in \mathbb{C}^{n-1}$, $r \in \mathbb{R}^{n-1}$, $Y \in \mathcal{SP}_n$, and where $p_{-s}(Y)$ is the power function defined earlier. This is a reasonable generalization since, for $s \in \mathbb{C}$, we may note that:

$$\begin{aligned} k_1\left(s, r, \begin{pmatrix} y^{-1} & 0 \\ 0 & y \end{pmatrix}\right) &= \int_{-\infty}^{\infty} p_{-s}\left(\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\right)\begin{bmatrix} 1 & 0 \\ u & 1 \end{bmatrix} e^{2\pi i r u}\, du \\ &= y^s \int_{-\infty}^{\infty} (y^2 + u^2)^{-s} e^{2\pi i r u}\, du \\ &= \begin{cases} 2\pi^s \Gamma(s)^{-1} |r|^{s-1/2} y^{1/2} K_{s-1/2}(2\pi |r| y) & \text{if } r \neq 0, \\ \Gamma(\frac{1}{2})\Gamma(s - \frac{1}{2})\Gamma(s)^{-1} y^{1-s} & \text{if } r = 0. \end{cases} \end{aligned}$$

The power function is an eigenfunction of the $GL_n(\mathbb{R})$-invariant differential operators, so we find that $k_{n-1}(s, r, Y)$ is also.

Unfortunately, it is difficult to get the required asymptotic results from this version of the Bessel function on $\mathcal{P}_n$. There is another generalization of the

K-Bessel function that will prove useful—one that is defined to point out similarities with the K-Bessel function as in (3). For $s \in \mathbb{C}^n$ and $A, B \in \mathscr{P}_n$, and if $d\mu_n$ is the $GL_n(\mathbb{R})$-invariant volume, define

$$K_n(s \mid A, B) = \int_{Y \in \mathscr{P}_n} p_{-s}(Y) e^{\operatorname{Tr}(AY + BY^{-1})} \, d\mu_n(Y) . \tag{5}$$

This function can also be extended to singular B with s suitably restricted to make sure the integral converges. It is easily seen that for $a, b > 0$ and $s \in C$

$$K_1(s \mid a, b) = \int_0^\infty y^{-1} e^{-(ay+by^{-1})} \frac{dy}{y} = 2\left(\frac{a}{b}\right)^{s/2} K_{-s}(2\sqrt{ab}) .$$

Since the K-Bessel function has functional equation $K_{-s}(y) = K_s(y)$, $K_1(s \mid a, b) = 2(\frac{a}{b})^{s/2} K_s(2\sqrt{ab})$.

We have now seen two generalizations of the K-Bessel function, and have seen that each does indeed reduce to the K-Bessel function. It should come as no surprise, then, that there is a relation between the functions defined in (4) and (5). First, we need the $\mathscr{P}_n$ version of the Γ function:

$$\Gamma_n(s) = \int_{Y \in \mathscr{P}_n} p_s(Y) e^{-\operatorname{Tr}(Y)} \, d\mu_n(Y)$$

for $s \in \mathbb{C}^n$ with $\operatorname{Re}(s_j + \cdots + s_n) > \frac{j-1}{2}$, $j = 1, \ldots, n$. Γ_n can now mean two different things, but which should be clear from context. It is also clear that this Γ_n is the ordinary Γ function for $n = 1$. We have the following, due to Bengtson [1] (also summarized in [11, Vol. II]).

Lemma 1. *For $s \in \mathbb{C}^{n-1}$, let $\hat{s} = (s_1 - \frac{1}{2}, s_2, \ldots, s_{n-1})$. If the coordinates of s are restricted to suitable half-planes, we have*

$$\Gamma_{n-1}(-s) k_{n-1}(s, r, Y) = \pi^{(n-1)/2} v^{(n-1)/2} K_{n-1}(\hat{s} \mid v^{1/(n-1)} W, \ v[\pi\, {}^T r])$$

if $Y = \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)} W \end{pmatrix}$.

Since $r \in \mathbb{R}^{n-1}$, $v[\pi\, {}^T r]$ is an $n-1$ square matrix, but it has rank 1. As previously indicated, the definition of $K_n(s \mid A, B)$ could be extended to singular B for suitable choices of s. The k-Bessel function is an eigenfunction of the $GL_n(\mathbb{R})$ invariant differential operators, and so we expect it will appear in Fourier expansions of automorphic forms. However, as $v \to \infty$, it is easier to see what happens if we use the integral representation (5) of the K-Bessel function as in the following:

Lemma 2. *For $A \in \mathscr{P}_n$, $b > 0$, and $s \in \mathbb{C}^n$ with $\operatorname{Re}(s_j + \cdots + s_n) < \frac{j-1}{2}$, $K_n\left(s \mid A, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right)$ approaches 0 exponentially as $b \to \infty$.*

Proof.

$$K_n\left(s \mid A, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right) = \int_{Y \in P_n} p_{-s}(Y) e^{-\operatorname{Tr}(AY + \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) Y^{-1})} \, d\mu_n(Y) .$$

Let $Y = I[T]$ with T upper triangular. Then $p_{-s}(Y) = \prod_{j=1}^n t_{jj}^{-r_j}$ with $r_j = 2(s_j + \cdots + s_n)$. Also $Y^{-1} = I[{}^T T^{-1}]$ and we will write

$$T^{-1} = \begin{pmatrix} t_{11}^{-1} & & t^{ij} \\ & \ddots & \\ 0 & & t_{nn}^{-1} \end{pmatrix} .$$

If a is the smallest eigenvalue of A, then

$$K_n\left(s \mid A, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right) \le 2^n \int_T e^{-\operatorname{Tr}(AI[T]+\left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)I[{}^TT^{-1}])} \prod_{j=1}^n t_{jj}^{\operatorname{Re} r_j - j} \prod_{1\le i\le j\le n} dt_{ij}$$
$$\le 2^n \int_T e^{-a\operatorname{Tr}(I[T]) - b\sum_{j=1}^n (t^{1j})^2} \prod_{j=1}^n t_{jj}^{\operatorname{Re} r_j - j} \prod_{1\le i\le j\le n} dt_{ij},$$

where the 2^n is the Jacobian of the change of variables and $t^{11} = t_{11}^{-1}$. Since $e^{-b\sum_{j=1}^n (t^{1j})^2} \le e^{-bt_{11}^{-2}}$, it follows that

$$K_n\left(s \mid A, \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}\right) \le 2^n \int_{t_{11}>0} e^{-at_{11}^2 - bt_{11}^{-2}} t_{11}^{\operatorname{Re} r_1 - 1}\, dt_{11}$$
$$\times \prod_{j=2}^n \int_{t_{jj}>0} e^{-at_{jj}^2} t_{jj}^{\operatorname{Re} r_j - j}\, dt_{jj} \prod_{1\le i<j\le n} \int_{t_{ij}\in\mathbb{R}} e^{-at_{ij}^2}\, dt_{ij}$$
$$\le 2^n \left(\frac{\pi}{a}\right)^{[n(n-1)]/4} \prod_{j=2}^n \left(\Gamma\left(\frac{j - \operatorname{Re} r_j - 1}{2}\right) a^{(\operatorname{Re} r_j - j + 1)/2}\right)$$
$$\times 2^{-n} \cdot 2 \left(\frac{a}{b}\right)^{(\operatorname{Re} r_1 - 1)/4} K_{(\operatorname{Re} r_1)/2}(2\sqrt{ab})$$
$$\le 2\pi^{[n(n-1)]/4} a^{-[n(n-1)]/2} \prod_{j=2}^n \left(\Gamma\left(\frac{j - \operatorname{Re} r_j - 1}{2}\right) a^{\operatorname{Re} r_j/2}\right)$$
$$\times \left(\frac{a}{b}\right)^{\operatorname{Re} r_1 - 1/4} K_{(\operatorname{Re} r_1)/2}(2\sqrt{ab}).$$

For the convergence of the Γ functions we need $\operatorname{Re}(s_j + \cdots + s_n) < \frac{j-1}{2}$ and since $K_s(y) \le \sqrt{\pi/2y}\, e^{-y}$ for $y > 0$ (see [5]) the lemma is proved.

Yet a third special function on $\mathscr{P}_n$ related to the K-Bessel function is the Whittaker function. This appears in the Fourier expansions of many authors including Bump [2] and Shalika [10], whose methods we will discuss in the next section. For $s \in \mathbb{C}^{n-1}$, $r \in \mathbb{R}^{n-1}$, and $Y \in \mathscr{S}P_n$ the Whittaker function can be defined by

$$\mathscr{W}(s, r, Y) = \int_{N\in\mathscr{N}} p_{-s}(Y^{-1}[{}^TN]) e^{2\pi i \sum_{j=1}^{n-1} r_j x_{j j+1}}\, dN \tag{6}$$

where $\mathscr{N}$ is the nilpotent subgroup of $\mathscr{P}_n$ of matrices N,

$$N = \begin{pmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{pmatrix},$$

and $dN = \prod_{1\le i<j\le n} dx_{ij}$. When $n = 2$ the Whittaker function virtually coincides with the K-Bessel function; when $n > 2$ the Whittaker function can be expressed as a Fourier transform of the k-Bessel function (4). For $s \in \mathbb{C}^{n-1}$,

$r \in \mathbb{R}^{n-1}$, and $Y \in \mathscr{S}P_n$:

$$W(s, r, Y) = \int_{x_{ij} \in \mathbb{R}} k_{n-1}\left(s, (r_1, 0, \ldots, 0), Y\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & x_{ij} \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}\right) \times e^{2\pi i \sum_{j=2}^{n-1} r_j x_{jj+1}} \prod_{2 \le i < j \le n} dx_{ij}.$$

By this, the k-Bessel functions are also Fourier transforms of the Whittaker functions. This enables one to relate Fourier expansions of automorphic forms for Γ_n in Whittaker functions to those involving k-Bessel and thus K-Bessel functions.

3. Fourier expansions of automorphic forms

As mentioned previously the Fourier expansions for Maass wave forms (2) involve the usual K-Bessel function. In particular, if $E_s(z)$ is the Eisenstein series

$$E_s(z) = \sum_{\gamma \in SL_2(\mathbb{Z})/\Gamma_\infty} \operatorname{Im}(\gamma z) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \frac{y^s}{|cz+d|^{2s}},$$

where Γ_∞ is the subgroup of $\Gamma = SL_2(\mathbb{Z})$ containing matrices $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, $m \in \mathbb{Z}$, and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ for $\gamma \in \Gamma$, it can be shown to have a Fourier expansion of the form

$$E_s(z) = y^s + y^{1-s}\frac{\Lambda(1-s)}{\Lambda(s)} + \frac{2}{\Lambda(s)}\sum_{n \neq 0} |n|^{s-1/2}\sigma_{1-2s}(n) y^{1/2} K_{s-1/2}(2\pi|n|y) e^{2\pi i n x},$$

where $\Lambda(s) = \pi^{-s}\Gamma(s)\zeta(2s)$. Many authors have generalized this to Eisenstein series for Γ_n and in particular for $GL_3(\mathbb{Z})$. For Γ_n let us consider a special type of the more general Eisenstein series introduced by Selberg [9]. For $s \in \mathbb{C}^{n-1}$ and $Y \in \mathscr{S}P_n$ define

$$E_n(s, Y) = \sum_{\gamma \in \Gamma_n/\Gamma_\infty} p_{-s}(Y[\gamma]), \tag{7}$$

where as before Γ_∞ is the subgroup of Γ_n of upper triangular matrices. This series converges for $\operatorname{Re}(s_j) > 1$, and has analytic continuation for each of the s_j variables (the ideas for the analytic continuation go back to Selberg as in [8, 9] and are summarized by Terras in [11]). Vinogradov and Takhtadzhyan [13] obtained Fourier expansions of $E_3(s, Y)$ involving products of the usual K-Bessel functions; Imai and Terras [4] and Terras [11] considered more general Eisenstein series on Γ_3 and obtained Fourier expansions with matrix k-Bessel and K-Bessel functions; Terras also discusses how to extend the Chowla-Selberg method to expand certain Eisenstein series on Γ_n in [12]; Bump [2] uses Whittaker functions in his Fourier expansions of Eisenstein series on Γ_3. Following these examples we expect to see Fourier expansions of automorphic forms on

Γ_n look like

$$f(Y) = a_0(v, W) + \sum_{\substack{m\in\mathbb{Z}^{n-1}\\ m\neq 0}} \sum_{g\in\Gamma_{n-1}/P} a_m v^{(n-1)/2} \times K_{n-1}(\hat{s}, v^{1/(n-1)}W[g], v[\pi\, {}^T m])e^{2\pi i\, {}^T xgm}, \tag{8}$$

where the v, W, and x come from the decomposition (1), P is a parabolic subgroup of Γ_{n-1} to be defined shortly, and $s \in \mathbb{C}^{n-1}$ in the argument of the K-Bessel functions is determined by the eigenvalues of the differential operators acting on f.

Recall that for Maass wave forms (automorphic forms for $n = 2$) the eigenvalues were written $\lambda = s(s-1)$. One reason for this is that if Δ is the Laplace operator on $\mathscr{S}P_2$, then $\Delta y^s = s(s-1)y^s$ and y^s is the power function on $\mathscr{S}P_2$. Since the Eisenstein series is a summation of the power function over the modular group and Δ is $GL_2(\mathbb{R})$-invariant, the eigenvalue corresponding to $E_s(z)$ is $s(s-1)$. If f is any Maass form and $\Delta f = \lambda f$, then λ can be written as $s(s-1)$ (the other solution of $s(s-1) = \lambda$ is $1-s$).

For $\mathscr{S}P_n$, we have the L_j as defined earlier determining the ring of differential operators, with L_2 being the Laplacian. Explicitly, in terms of the decomposition (1), we have

$$\Delta_n = \frac{n-1}{n}v^2\frac{\partial^2}{\partial v^2} - \frac{1}{n}v\frac{\partial}{\partial v} + v^{n/(n-1)}W\left[\frac{\partial}{\partial x_j}\right] + \Delta_{n-1}$$

inductively expressing the Laplacian on $\mathscr{S}P_n$. Then the eigenvalue of Δ_n corresponding to the power function, $p_{-s}(Y)$, and hence to the Eisenstein series, $E_n(s, Y)$, (7), is

$$\lambda = \frac{n-1}{n}(s_1+\xi_1)\left(s_1 - 1 + \xi_1 - \frac{1}{n-1}\right) + \frac{n-2}{n-1}(s_2+\xi_2)\left(s_2 - 1 + \xi_2 - \frac{1}{n-2}\right) + \cdots + \frac{1}{2}s_{n-1}(s_{n-1}-1),$$

where for convenience we define

$$\xi_j = \frac{1}{n-j}\sum_{k=j+1}^{n-1}(n-k)s_k, \qquad j = 1, 2, \ldots, n-2, \ \xi_{n-2} = 0.$$

For any automorphic form f we will have n differential equations $L_j f = \lambda_j f$. Then we can find $s \in \mathbb{C}^{n-1}$ satisfying the various relations determined by the λ_j when $f = E_n(s, Y)$, in particular the one for the eigenvalue of the Laplacian above. Since the k-Bessel function (4) is defined by the power function we expect the Fourier expansion of f with eigenvalues determined by $s \in \mathbb{C}^n$ to involve $k_{n-1}(s, r, Y)$.

Since $f(Y)$ is invariant under matrices of the form $\begin{pmatrix} 1 & {}^T b \\ 0 & I\end{pmatrix}$ with $b \in \mathbb{Z}^{n-1}$, there is a Fourier expansion

$$f(Y) = \sum_{N\in\mathbb{Z}^{n-1}} a_N(v, W)e^{2\pi i\, {}^T x N}, \tag{9}$$

again decomposing Y as in (1). The a_N can be expressed as

$$a_N(v, W) = \int_0^1 \int_0^1 \cdots \int_0^1 f(Y) e^{-2\pi i\, {}^T x N}\, dx .$$

Lemma 3. *For* $g \in \Gamma_{n-1}$, $a_{gN}(v, W) = a_N(v, W[g])$.

Proof. If $g \in \Gamma_{n-1}$, then $\begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} \in \Gamma_n$, and

$$\begin{aligned} Y\begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} &= \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & g \end{bmatrix} \\ &= \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x\, g \\ 0 & g \end{bmatrix} \\ &= \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[g] \end{pmatrix} \begin{bmatrix} 1 & {}^T x\, g \\ 0 & I \end{bmatrix}. \end{aligned}$$

Then, with $a_N(v, W)$ as defined above,

$$\begin{aligned} a_N(v, W[g]) &= \int f\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[g] \end{pmatrix} \begin{bmatrix} 1 & {}^T x \\ 0 & I \end{bmatrix} \right) e^{-2\pi i\, {}^T x N}\, dx \\ &= \int f\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x\, g^{-1} \\ 0 & I \end{bmatrix} \right) e^{-2\pi i {}^T x N}\, dx \\ &= \int f\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/n-1}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x \\ 0 & g \end{bmatrix} \right) e^{-2\pi i\, {}^T x N}\, dx \\ &= \int f\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/n-1}W \end{pmatrix} \begin{bmatrix} 1 & {}^T x\, g^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & g^{-1} \end{bmatrix} \right) e^{-2\pi i\, {}^T x N}\, dx \\ &= \int f(Y) e^{-2\pi i\, {}^T u g N}\, du \\ &= a_{gN}(v, W), \end{aligned}$$

where we made the change of variables $x = {}^T g u$.

If e_1 is the first standard unit vector for $\mathbb{R}^{n-1}$, let $a_j(v, W)$ be short for $a_{je_1}(v, W)$. Then, by Lemma 3, $a_N(v, W) = a_j(v, W[g])$ for some g such that $gje_1 = N$ (j will be the gcd of N). Then, define P to be the parabolic subgroup of Γ_{n-1} consisting of matrices of the form $\begin{pmatrix} \pm 1 & * \\ 0 & g' \end{pmatrix}$ with $g' \in \Gamma_{n-2}$. Note that for $g \in P$, $ge_1 = \pm e_1$. Thus the series (9) becomes

$$f(Y) = a_0(v, W) + \sum_{j \neq 0} \sum_{g \in \Gamma_{n-1}/P} a_j(v, W[g]) e^{2\pi i\, {}^T x j e_1} . \tag{10}$$

In [2] Bump shows how to translate Shalika's adelic version of Fourier expansions of automorphic forms into the classical approach we are trying to follow, at least for $GL_3(\mathbb{R})$. Extending these ideas to $GL_n(\mathbb{R})$ yields

$$\begin{aligned} f(Y) = a_0(v, W) + \sum_{\substack{m \in \mathbb{Z}^{n-1} \\ m \neq 0}} \sum_{g \in \Gamma_{n-1}/P} & a_m \\ & \times \mathscr{W}\left(s, m, \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[g] \end{pmatrix} \right) e^{2\pi i\, {}^T x g m}, \end{aligned}$$

where $\mathscr{W}(s, m, Y)$ is the Whittaker function (6). Since the k-Bessel functions are Fourier transforms of the Whittaker functions, the Fourier expansion of f can also be written

$$f(Y) = a_0(v, W) + \sum_{\substack{m\in\mathbb{Z}^{n-1}\\ m\neq 0}} \sum_{g\in\Gamma_{n-1}/P} a_m \times k_{n-1}\left(s, m, \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[g]\end{pmatrix}\right) e^{2\pi i\, {}^T x g m}. \tag{11}$$

See [11, Vol. II] for details. Also note that the a_m here are different from above.

Note that in the expansion (9) the a_n would be

$$a_n(v, W) = \sum_{\substack{n\in\mathbb{Z}^{n-1}\\ \gcd(m)=\gcd(n)}} \sum_{\substack{h\in\Gamma_{n-1}/P\\ hm=N}} a_m k_{n-1}\left(s, m, \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[h]\end{pmatrix}\right)$$

and this satisfies Lemma 3 since

$$\begin{aligned} a_N(v, W[g]) \sum_m \sum_{\substack{h\in\Gamma_{n-1}/P\\ hm=N}} & a_m k_{n-1}\left(s, m, \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[gh]\end{pmatrix}\right) \\ = \sum_m \sum_{\substack{gh\in\Gamma_{n-1}/P\\ ghm=gN}} & a_m k_{n-1}\left(s, m, \begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W[gh]\end{pmatrix}\right) \\ = a_{gN}(v, W) & \quad \text{since } \gcd(gN) = \gcd(N). \end{aligned}$$

Finally, by virtue of Lemma 1, we can summarize the results so far as

Theorem 1. *An automorphic form f for Γ_n, with eigenvalues determined by $s \in \mathbb{C}^n$, has Fourier expansion*

$$f(Y) = a_0(v, W) + \sum_{\substack{m\in\mathbb{Z}^{n-1}\\ m\neq 0}} \sum_{g\in\Gamma_{n-1}/P} a_m v^{(n-1)/2} \times K_{n-1}(\hat{s} \mid v^{1/(n-1)}W[g], v[\pi\, {}^T m]) e^{2\pi i\, {}^T x g m}. \tag{12}$$

Now it is time to examine the a_0 term more closely.

4. Automorphic forms on Γ_n and Γ_{n-1}

Once more we look to the Eisenstein series to provide an example. Recall (7):

$$E_n(s, Y) = \sum_{\gamma\in\Gamma_n/\Gamma_\infty} p_{-s}(Y[\gamma]).$$

Since the power function is invariant under the action of upper triangular matrices,

$$p_{-s}(Y) = p_{-s}\left(\begin{pmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)}W\end{pmatrix}\right) = v^{s_1+\xi_1} p_{-s'}(W),$$

where if ${}^T s = (s_1, s_2, \dots, s_{n-1})$, then ${}^T s' = (s_2, \dots, s_{n-1})$, and as before $\xi_1 = \frac{1}{n-1}\sum_{k=2}^{n-1}(n-k)s_k$.

If we write $\gamma \in \Gamma_n$ as $\gamma = \left(\begin{smallmatrix} a & {}^T b \\ c & D \end{smallmatrix}\right)$, then

$$Y[\gamma] = \begin{pmatrix} v^{-1}[a + {}^T x\, c] + v^{1/(n-1)} W[c] & 0 \\ 0 & v^{-1}[{}^T b + {}^T x\, D] + v^{1/(n-1)} W[D] - v'[{}^T x'] \end{pmatrix} \times \begin{bmatrix} 1 & {}^T x' \\ 0 & I \end{bmatrix}$$

where $v' = v^{-1}[a + {}^T x\, c] + v^{1/(n-1)} W[c]$ and

$${}^T x' = \frac{(a + {}^T x\, c) v^{-1}({}^T b + {}^T x\, D) + {}^T c v^{1/(n-1)} W D}{v'}.$$

Then

$$p_{-s}(Y[\gamma]) = (v')^{-s_1-\xi_1} p_{-s'}((vv')^{1/(n-1)} \cdot (vv')^{-1} W[(a + {}^T x c)D - c({}^T b + {}^T x\, D)])$$

and

$$E_n(s, Y) = v^{s_1+\xi_1} \sum_{\gamma \in \Gamma_n/\Gamma_\infty} ((a + {}^T x\, c)^2 + v^{n/(n-1)} W[c])^{-s_1+s_3+2s_4+\cdots+(n-3)s_{n-1}} \times p_{-s'}(W[(a + {}^T x\, c)D - c({}^T b + {}^T x\, D)]).$$

As $v \to \infty$, all the terms in the series will approach 0 because of the $v^{n/(n-1)}$ term, if s is chosen to make the exponent negative, except for when $c = 0$. In this case a must be ± 1. Then as $v \to \infty$,

$$E_n(s, Y) \sim v^{s_1+\xi_1} \sum_{D \in \Gamma_{n-1}/\Gamma_\infty} p_{-s'}(W[D]).$$

So

$$E_n(s, Y) \sim v^{s_1+\xi_1} E_{n-1}(s', W).$$

This is another indication that we should be able to obtain some sort of analogue to the ϕ-operator previously mentioned. We begin by stating the following theorem. The proof is made somewhat more difficult than the case of Siegel's modular forms since we are not dealing with analytic automorphic forms.

Theorem 2. *If $f(Y)$ is an automorphic form for Γ_n with eigenvalues determined by $s \in \mathbb{C}^n$, then define*

$$f|\phi(W) = \lim_{v \to \infty} v^{-s_1-\xi_1} f(Y),$$

where $Y = \left(\begin{smallmatrix} v^{-1} & 0 \\ 0 & v^{1/(n-1)} W \end{smallmatrix}\right) \left[\begin{smallmatrix} 1 & {}^T x \\ 0 & I \end{smallmatrix}\right]$. Then $f|\phi(W)$ is an automorphic form for Γ_{n-1}.

Proof. Consider the Fourier expansion (12). We have K-Bessel functions $K_{n-1}(\hat{s} \mid A, B)$, where $A = v^{1/(n-1)} W[g]$, and $B = v[\pi\, {}^T m]$. If ${}^T m = (m_1, \ldots, m_{n-1})$ and $m_1 \neq 0$, then we have

$$B = \pi^2 v (m_i m_j)_{ij} = \pi^2 v \begin{pmatrix} m_1^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{bmatrix} 1 \frac{m_2}{m_1} & \cdots & \frac{m_{n-1}}{m_1} \\ 0 & & I \end{bmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} [c],$$

where $b = \pi^2 v m_1^2$ and c is as above. From definition (5),

$$\begin{aligned} K_{n-1}(\hat{s} \mid A, B) &= \int_{Y \in \mathscr{P}_{n-1}} p_{-\hat{s}}(Y) e^{-\operatorname{Tr}(AY + \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right)[c] Y^{-1})}\, d\mu_{n-1} \\ &= \int_{Y \in \mathscr{P}_{n-1}} p_{-\hat{s}}(Y) e^{-\operatorname{Tr}(AY + \left(\begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix}\right) Y[{}^T c])}\, d\mu_{n-1}(Y). \end{aligned}$$

Let $U = Y[c^{-1}]$. Since the power function is invariant under the action of upper triangular matrices,

$$K_{n-1}(\hat{s} \mid A, B) = \int_{Y\in\mathscr{P}_{n-1}} p_{-\hat{s}}(U)e^{-\operatorname{Tr}(A[^Tc]U+\left(\begin{smallmatrix} b & 0\\ 0 & 0\end{smallmatrix}\right)U^{-1})}\, d\mu_{n-1}(U)$$
$$= K_{n-1}\left(\hat{s} \mid A[^Tc],\ \begin{pmatrix} b & 0\\ 0 & 0\end{pmatrix}\right).$$

If a is the smallest eigenvalue of A, then both $a \to \infty$ and $b \to \infty$ as $v \to \infty$. By Lemma 2, $K_{n-1}(\hat{s}|A, B) \to 0$ exponentially. Thus all the terms in the expansion (12) with $m_1 \neq 0$ vanish as $v \to \infty$. The sum (12) is over all $m \in \mathbb{Z}^{n-1}$, $m \neq 0$, so if $m_1 = 0$ there is a smallest j such that $m_j \neq 0$. Then if we write A and B as

$$A = \begin{pmatrix} A_1 & 0\\ 0 & A_2\end{pmatrix}\begin{bmatrix} I & 0\\ ^Tc & I\end{bmatrix}, \qquad B = \begin{pmatrix} 0 & 0\\ 0 & B_2\end{pmatrix},$$

where $A_1 \in \mathscr{P}_{j-1}$, $A_2 \in \mathscr{P}_{n-j}$, and $B_2 = \pi^2 v\left(\begin{smallmatrix} m_j^2 & \cdots\\ \vdots & \ddots\end{smallmatrix}\right)$, a formula of Terras [11, Vol. II, p. 54] shows that

$$K_{n-1}(\hat{s}|A, B) = \pi^{[(j-1)(n-j)]/2}|A_2|^{(1-j)/2}p_{-\sigma_1}(A_1^{-1})\Gamma_{j-1}(-\sigma_1)K_{n-j}(\sigma_2|A_2, B_2).$$

Here $\Gamma_{j-1}(-\sigma_1)$ is the gamma function on $\mathscr{P}_{j-1}$ and σ_1 and σ_2 are determined from $\hat{s}$. Now Lemma 2 can be applied to $K_{n-j}(\sigma_2|A_2, B_2)$ and we have shown that as $v \to \infty$ all terms in the expansion (12) vanish except $a_0(v, W)$. So as $v \to \infty$, $f(Y) \sim a_0(v, W)$. To prove the theorem we must show that $f|\phi(W) = \lim_{v\to\infty} v^{-s_1-\xi_1}a_0(v, W)$ satisfies the three properties in the definition of an automorphic form:

(i) $f|\phi(W)$ is an eigenfunction for all the $GL_{n-1}(\mathbb{R})$-invariant differential operators: Since $a_0(v, W)$ contains no x_j terms, the operators L_j reduce to $L_j = L_{j,v} + L_{j,W}$, where $L_{j,v}$ operates on v alone and the $L_{j,W}$ will be the corresponding $GL_{n-1}(\mathbb{R})$-invariant operator on $W \in \mathscr{S}P_{n-1}$. By virtue of this, $f|\phi(W)$ is an eigenfunction for all the $G_{n-1}(\mathbb{R})$-invariant differential operators.

(ii) For all $\gamma \in \Gamma_{n-1}$, $W \in \mathscr{S}P_{n-1}$, and $f|\phi(W[\gamma]) = f|\phi(W)$: By Lemma 3, $a_N(v, W[g]) = g_{gN}(v, W)$ for $g \in \Gamma_{n-1}$. Letting $n = 0$ we get $a_0(v, W[g]) = a_0(v, W)$ and we have invariance of $f|\phi(W)$ under Γ_{n-1}.

(iii) $|f|\phi(W)| \leq C|p_{-s'}(W)|$ for some $C > 0$, $s' \in \mathbb{C}^{n-2}$ as $|W_j| \to \infty$: Since f is an automorphic form for Γ_n we start with $|f(Y)| \leq C|p_{-s}(Y)|$ for some $C > 0$, $s \in \mathbb{C}^{n-1}$ as $|Y_j| \to \infty$.

$$a_0(v, W) = \int_0^1\int_0^1\cdots\int_0^1 f(Y)\,dx,$$
$$|a_0(v, W)| \leq \int |f(Y)|\,dx \leq C\int |p_{-s}(Y)|\,dx$$
$$\leq C\int |v^{s_1+\xi_1}|\,|p_{-s'}(W)|\,dx \leq C|v^{s_1+\xi_1}|\,|p_{-s'}(W)|.$$

So we see that $|f|\phi(W)| \leq C|p_{-s'}(W)|$ and we have completed the proof of Theorem 2.

A cusp form for Γ_n is an automorphic form satisfying

$$\int_{X\in(\mathbb{R}/\mathbb{Z})^{j\times(n-j)}} f\left(Y\begin{bmatrix} I & X \\ 0 & I\end{bmatrix}\right) dX = 0; \qquad j = 1, 2, \dots, n-1.$$

When $j = 1$ this integral becomes $a_0(v, W) = 0$, so for f a cusp form on Γ_n, $f|\phi$ is identically 0.

It is now possible to rewrite equation (12) as:

$$f(Y) = a(s, v)f|\phi(W) + \sum_{\substack{m\in\mathbb{Z}^{n-1}\\ m\neq 0}} \sum_{g\in\Gamma_{n-1}/P} a_m v^{(n-1)/2} \tag{13}$$
$$\times K_{n-1}(\hat{s} \mid v^{1/(n-1)}W[g], v[\pi\, {}^T m])e^{2\pi i\, {}^T x g m}.$$

Theorem 2 should be quite useful in the study of automorphic forms for Γ_n if the corresponding ideas for Siegel's modular forms are any indication. In [7], Maass proves several results on the space of Siegel's modular forms using inductive arguments aided by Siegel's ϕ-operator. We should be able to prove analogous results in some cases for our automorphic forms for Γ_n. As previously mentioned, Theorem 2 goes hand in hand with the behavior of the fundamental domain, $\mathscr{F}_n$, as $v \to \infty$. Basically, this will allow us to generalize a theorem of Maass [6] on the integral of a product of two automorphic forms. The details of this will appear later; here we will just summarize the application of Theorem 2 to extending Maass' result.

5. The Maass-Selberg relations

In [6] Maass proved the following:

If f and g are Maass wave forms for $SL_2(\mathbb{Z})$ with Fourier expansions

$$f(z) = \sum_{m\in\mathbb{Z}} a_m(y)e^{2\pi i m x}, \qquad g(z) = \sum_{m\in\mathbb{Z}} b_m(y)e^{2\pi i m x}$$

and D_A is the truncated fundamental domain $D_A = \{z \in H \mid |z| \geq 1, |x| \leq \frac{1}{2}, y \leq A\}$, then

$$\int_{D_A} (f\Delta g - g\Delta f)\frac{dx\,dy}{y^2} = \sum_{m\in\mathbb{Z}} (a_m(A)b'_{-m}(A) - a'_m(A)b_{-m}(A)). \tag{14}$$

The proof makes use of the invariance of f and g under $SL_2(\mathbb{Z})$ and an old friend: Green's Theorem. It can then be shown that if f and g have the same eigenvalue, then $a_m b'_{-m} - a'_m b_{-m} = 0$ for $m \neq 0$. Therefore $a_0 b'_0 - a'_0 b_0 = 0$ and this leads to the one-dimensionality of the space of constant terms in the Fourier expansions of Maass wave forms with eigenvalue λ. This of course is trivial in the case of holomorphic modular forms. From this result we can see that any nonholomorphic form is the sum of an Eisenstein series and a cusp form (a form with constant term 0) just as in the case of the holomorphic forms. A second place the integral (14) is useful is in the Selberg trace formula. The parabolic term in that can be transformed into something involving $\int_D E_s(z)E_{\bar{s}}(z)\frac{dx\,dy}{y^2}$. If we let $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$ and $\varphi(s) = \frac{\Lambda(2(1-s))}{\Lambda(2s)}$ then

$$\int_{D_A} E_s(z)E_{\bar{s}}(z)\frac{dx\,dy}{y^2} = 2\log A - \frac{\varphi'(s)}{\varphi(s)} + \frac{\varphi(\bar{s})A^{2ir} - \varphi(s)A^{-2ir}}{2ir} + o(1)$$

for $s = \frac{1}{2} + ir$. This enables one to evaluate one of the most difficult terms in the Selberg trace formula. For similar work on Γ_n for $n > 2$ we will need the corresponding version of (14), an explicit version of the Maass-Selberg relations on $GL_n(\mathbb{Z})$. This was begun in [3], with much of the foundation prepared, but something was missing. That something is the ϕ-operator from the previous section.

First, let us summarize the necessary preliminaries. The Laplace operator on $\mathscr{S}P_n$ can be expressed as

$$\Delta_n = \frac{n-1}{n} v^2 \frac{\partial^2}{\partial v^2} - \frac{1}{n} v \frac{\partial}{\partial v} + v^{n/(n-1)} W \left[\frac{\partial}{\partial x_j} \right] + \Delta_{n-1}.$$

Also, the Γ_n-invariant volume element is

$$d\mu_n = v^{(-n/2)-1} dv\, dx\, d\mu_{n-1}, \quad \text{where } dx = dx_1 dx_2 \cdots dx_{n-1}.$$

Now, define $\mathscr{F}_n^*$ to be

$$\mathscr{F}_n^* = \bigcup_{\gamma \in \mathscr{D}_n} \mathscr{F}_n[\gamma]$$

with $\mathscr{D}_n$ the set of diagonal matrices in Γ_n, i.e., diagonal matrices with entries ± 1. Clearly $\mathscr{F}_n$ is a subset of $\mathscr{F}_n^*$ and in fact it is easily seen that the volume of $\mathscr{F}_n^* = 2^{n-1}\operatorname{vol}(\mathscr{F}_n)$. Then, letting $\mathscr{F}_A$ be the truncated fundamental domain $\mathscr{F}_A = \{Y \in \mathscr{F}_n \mid v \le A\}$, if f and g are automorphic forms on Γ_n with eigenvalues $\Delta_n f = \lambda_f f$, and $\Delta_n g = \lambda_g g$, we have

$$\begin{aligned} 2^{n-1}(\lambda_g - \lambda_f) \int_{\mathscr{F}_A} f(Y) g(Y)\, d\mu_n(Y) &= \int_{\mathscr{F}_A^*} (f \Delta_n g - g \Delta_n f)\, d\mu_n \\ &= \int_{\substack{v=A \\ Y \in \mathscr{F}_n^*}} v^{1-n/2} \left(f \frac{\partial g}{\partial v} - g \frac{\partial f}{\partial v} \right) dx\, d\mu_{n-1}, \end{aligned}$$

where we have first changed from the integral over the domain to an integral over its surface by the general form of Stokes' Theorem and then used the modular invariance to cancel all but the integral on the surface $v = A$. Then as $A \to \infty$ this last integral is asymptotic to

$$A^{1-n/2} \left(a(s, A) \frac{\partial b}{\partial v}(s', A) - b(s', A) \frac{\partial a}{\partial b}(s, A) \right) \times \int_{\mathscr{F}_{n-1}^*} (f \mid \phi)(g \mid \phi)\, d\mu_{n-1}.$$

We now have the inductive step in going from the Maass-Selberg relations for Γ_{n-1} to the Maass-Selberg relations for Γ_n. We also have a big mess, but we should see some simplification when we put all the terms together. Also, if f and g are Eisenstein series of the form (10) with $s = (\frac{1}{2} + ir_1, \dots, \frac{1}{2} + ir_{n-1})$ and $s' = \bar{s}$, then more simplification will occur; we will also know a and b explicitly (they should be in terms of functions like $\Lambda(s)$).

The ϕ-operator used here may also help in answering certain questions about cusp forms for Γ_n. Clearly, by Theorem 2, if f is a cusp form then $f|\phi = 0$, but one can see that the converse is not true by examining the Eisenstein series (for $GL_4(\mathbb{Z})$)

$$E(s, f_1, f_2 \mid Y) = \sum_{g \in \Gamma_4/P(2,2)} |a_1(Y[g])|^{-s} f_1 \left(\frac{a_1(Y[g])}{|a_1(Y[g])|^{1/2}} \right) f_2 \left(\frac{a_2(Y[g])}{|a_2(Y[g])|^{1/2}} \right)$$

where f_1 and f_2 are Maass cusp forms (for $SL_2(\mathbb{Z})$) and

$$Y = \begin{pmatrix} a_1(Y) & 0 \\ 0 & a_2(Y) \end{pmatrix} \begin{pmatrix} I_2 & X \\ 0 & I_2 \end{pmatrix}.$$

If we iterate the partial Iwasawa decomposition (1) we get

$$Y = \begin{pmatrix} y_1^{-1} & 0 & 0 & 0 \\ 0 & y_1^{1/3}y_2^{-1} & 0 & 0 \\ 0 & 0 & y_1^{1/3}y_2^{1/2}y_3^{-1} & 0 \\ 0 & 0 & 0 & y_1^{1/3}y_2^{1/2}y_3 \end{pmatrix} \begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then we see that for $\operatorname{Re} s$ sufficiently large

$$E(s, f_1, f_2 \mid Y) = (y_1^{2/3}y_2)^s f_1(z_1) f_2(z_2) + \sum_{g \neq I} (\ ),$$

where $z_1 = x_{12} + iy_1^{2/3}y_2^{-1/2}$, $z_2 = x_{34} + iy_3$. Write $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, D are 2×2 integral matrices. Then $a_1(Y[g]) = a_1(Y)[A + XC] + a_2(Y)[C]$. Then for $\operatorname{Re} s > 0$, as $y_1 \to \infty$, the $|a_1(Y[g])|^{-s}$ term in the series goes to 0 unless $C = 0$. Since $g \in \Gamma_4/P(2, 2)$, if $C = 0$ take $g = I$. So we see that $E(s, f_1, f_2 \mid Y) \sim (y_1^{2/3}y_2)^s f_1(z_1) f_2(z_2)$ as $y_1 \to \infty$. If $y_2 \leq m$, since f_1 is a cusp form, $E(s, f_1, f_2 \mid Y) \to 0$ as $y_1 \to \infty$, and so $E \mid \phi \equiv 0$. (This example was suggested by the referee.) This example also shows that there ought to be another sort of ϕ-operator, call it ϕ_2, associated with the maximal parabolic $P(2, 2)$. This operator should send the Eisenstein series above to $f_1 f_2$. In general there should be one of these types of operators for each cusp. After working out the details of this perhaps one could obtain information on Fourier coefficients by comparing ways of sending a form on $GL_n(\mathbb{Z})$ to one on $GL_2(\mathbb{Z})$.

Instead of decomposing $Y \in \mathscr{S}P_n$ as (1) use

$$Y = D \begin{bmatrix} 1 & & x_{ij} \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \tag{15}$$

with D a diagonal matrix (the first entry would be v^{-1}, and so on). This can be obtained by using (1) and then repeating for $W \in \mathscr{S}P_{n-1}$, and continuing to repeat this until we get (15). Clearly the ϕ-operator will be compatible with this process, and it should also be possible to say something similar about the k-Bessel functions.

Most of the work on automorphic forms for $GL_n(\mathbb{Z})$ makes use of Fourier expansions in Whittaker functions as in [2]. It would be nice to do the ϕ-operator result using these Fourier expansions as well, but the integrals defining Whittaker functions seem to have problems with convergence, as do those for the k-Bessel functions. This was what made it necessary to introduce the second type of K-Bessel function. Perhaps the problems with Whittaker functions could be handled in the same way.

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